ON CERTAIN HOMOMORPHISMS OF QUOTIENTS OF GROUP ALGEBRAS

BY

KAREL DE LEEUW AND YITZHAK KATZNELSON

ABSTRACT

Let A be the algebra of functions on the circle group $T = \{z : |z| = 1\}$ having absolutely convergent Fourier series. For a subset E of A, the algebra of restrictions to E of functions in A is denoted by $A(E)$. It is shown that for sets that are "thick" in one of several senses, algebra homomorphisms of the $A(E)$ having norm 1 must arise from point mappings having a certain amount of rigidity.

Introduction. Let T be the circle group $\{z : |z| = 1\}$ and A the algebra of those functions on T having absolutely convergent Fourier series. A theorem of Beurling and Helson (see [1]) states that the only algebra automorphisms of A arise from a rigid motion of the circle T, that is, a rotation of T, or a reflection of T followed by a rotation.

If E is a closed subset of T, we denote by $A(E)$ the algebra of restrictions to E of the functions in A. Inspired by the result of Beurling and Helson one may inquire whether an algebra isomorphism of $A(E_1)$ and $A(E_2)$ must arise from a point mapping of E_1 and E_2 having a certain amount of rigidity. If E_1 and E_2 are intervals, this is indeed the case, as can be established by a modification of the argument of [1]. However, this technique yields no information for more general sets.

As a step in the direction of identifying algebra homomorphisms in the case of arbitrary closed subsets of T, we here investigate isomorphisms *of norm* 1. Although the results are not complete, we are able to show for a large class of closed sets, which are "thick" in one of several senses, that such an isomorphism must be induced by an affine point mapping.

A specialization of one of our results is the fact that if E_1 and E_2 are Cantor type sets with constant ratio of dissection, $A(E_1)$ and $A(E_2)$ cannot be isometrically isomorphic unless the ratios are the same. It seems likely that this continues to hold for algebra isomorphisms that are not necessarily isometric, but our methods shed no light on this problem.

1. Notations and statement of results.

We denote by T the circle group and by Z the group of integers. A is the Banach algebra of all continuous functions on T having an absolutely convergent Fourier series, the norm being defined by

Received January 19, 1964.

$$\| \sum a_n e^{int} \|_A = \sum | a_n | . \tag{1.1}$$

The dual space of A is known to be the space of all pseudomeasures on T, i.e. distributions on T having bounded Fourier coefficients. For $f \in A$ and $S \in A^*$ we put

$$\hat{f}(n) = \frac{1}{2\pi} \int f(t) e^{-int} dt, \quad \hat{S}(n) = < e^{-int}, S > ,$$

and since

$$f(t) = \sum \hat{f}(n) e^{int}$$

we have

$$< f, S > = \sum_{-\infty}^{\infty} \hat{f}(-n) \hat{S}(n), \tag{1.2}$$

and

$$\| S \|_{A^*} = \sup_n | \hat{S}(n) | . \tag{1.3}$$

(1.1) and (1.3) permit us to identify A and A^* with l^1 and l^∞ respectively.

Let $E \subset T$ be closed. We denote by $I(E)$ the ideal of all the functions in A that vanish on E, by $N(E)$ the space of all pseudo-measures orthogonal to $I(E)$ and by $A(E)$ the quotient algebra $A/I(E)$. $A(E)$ can be realized as the algebra of continuous functions on E which are restrictions to E of functions in A. Since $I(E)$ is closed, $A(E)$ is canonically a Banach algebra, its dual being $N(E)$.

Suppose that E and F are closed subsets of T and that

$$H : A(F) \to A(E)$$

is an algebraic homomorphism of $A(F)$ into $A(E)$. The sets E and F can be canonically identified with the maximal ideal spaces of $A(E)$ and $A(F)$ respectively, so by a familiar argument (see [2] p. 76) there is a continuous ϕ from E to F such that, for $f \in A(F)$,

$$H(f) = f \circ \phi. \tag{1.4}$$

DEFINITION. : ϕ is *affine* on E if there exists an integer n and a complex number c of modulus 1 such that

$$\phi(e^{it}) = c e^{int}, \ e^{it} \in E.$$

ϕ will be called *generalized affine* on E if there exists a multiplicative mapping $\chi : T \to T$ and a complex number c of modulus 1 such that

$$\phi(e^{it}) = c\chi(e^{it}), \ e^{it} \in E.$$

We assume throughout this paper $\| H \| = 1$ and prove the following results:

THEOREM 1.1 : *ϕ is always generalized affine.*

THEOREM 1.2 : *If E is a basis ϕ is affine.*

By the definition of the norm in $A(E)$ for every $f \in A(E)$ and $\varepsilon > 0$ there exists $g \in A$, $g \equiv f$ on E such that

$$\| g \|_A < \|f\|_{A(E)} + \varepsilon.$$

We shall say that E has *the extension property* if for every $f \in A(E)$ there exists $g \in A$, $g \equiv f$ on E and

$$\| g \|_A = \|f\|_{A(E)}.$$

E has the *restricted extension property* if norm preserving extension is required only for $f \in A(E)$ of norm 1 and modulus 1.

THEOREM 1.3. *If E has the restricted extension property, ϕ is affine.*

A pseudo-function is an element of A^* with Fourier coefficients tending to zero at infinity.

DEFINITION: E is an M^* set if $N(E)$ contains non-trivial pseudo-functions. E is a UM^* set (uniformly M^*) if $E \cap J$ is either empty or M^* set for every open interval J.

THEOREM 1.4. *If E is UM^* it has the extension property.*

COROLLARY. *If E is UM^*, ϕ is affine.*

THEOREM 1.5. *Suppose that for some $\varepsilon > 0$ there exists a positive measure μ of total mass 1 with arbitrarily small support contained in E such that*

$$\limsup_{|n| \to \infty} |\hat{\mu}(n)| < 1 - \varepsilon.$$

Then E has the restricted extension property

COROLLARY. *Under the condition of Theorem 1.5, ϕ is affine.*

Note that the condition of Theorem 1.5 is of a different nature than that of Theorem 1.4. A UM^* set can not be too thin at any of its points while the condition of Theorem 1.5 just implies thickness somewhere.

2. **Norm preserving extensions and affine mappings.** The proofs of Theorems 1.1 and 1.3 depend essentially on the following well known Lemma:

LEMMA 2.1. *Let G be a locally compact abelian group, ψ a continuous, positive positive-definite function on G, $\psi(1) = 1$. Then*

$$G_\psi = \{\sigma;\ \sigma \in G,\ |\psi(\sigma)| = 1\} \tag{2.1}$$

is a closed subgroup of G and on it ψ is multiplicative.

Proof. $\psi = \hat{\mu}$ where μ denotes a positive measure on $\hat{G}$. Now $|\psi(\sigma)| = 1$ if and only if σ, considered as character on $\hat{G}$, is constant (hence $= \psi(\sigma)$) on the support of μ.

Let us return now to the situation of §1. We have $\phi\colon E \to F$ such that for $f \in A(F), f \circ \phi \in A(E)$ and

$$\|f \circ \phi\|_{A(E)} \leqq \|f\|_{A(F)}. \tag{2.2}$$

Taking $f(e^{it}) = e^{it}$ we obtain $f \circ \phi = \phi$, hence $\phi \in A(E)$ and $\|\phi\| \leqq 1$; and since $|\phi| = 1$, $\|\phi\|_{A(E)} = 1$.

PROPOSITION 2.2. *Suppose that ϕ has a norm preserving extension. Then ϕ must be affine.*

Proof. By rotating E and F we may assume that $\phi(1) = 1$. Let ψ be a norm preserving extension of ϕ, that is

$$\psi(e^{it}) = \sum \hat{\psi}(n) e^{int}$$

satisfying

$$\begin{cases} \psi(e^{it}) \equiv \phi(e^{it}) \text{ for } e^{it} \in E \\ \sum |\hat{\psi}(n)| = 1. \end{cases} \tag{2.3}$$

Since $\psi(1) = \phi(1) = 1$, that is

$$\sum \hat{\psi}(n) = \sum |\hat{\psi}(n)|, \tag{2.4}$$

$\hat{\psi}(n) \geqq 0$ for all n and ψ is positive definite. T_ψ, defined by (2.1), is a *closed* subgroup of T, *containing* E, on which ψ is multiplicative; hence, for some n, $\psi(e^{it}) = e^{int}$ on T_ψ and the theorem is proved.

As an immediate corollary to Proposition 2.2 we obtain Theorem 1.3.

3. **Generalized norm preserving extensions.** Let E and F be finite sets containing N linearly independent points each. Let ϕ be any mapping of E onto F. Then by Kronecker's theorem, $f \to f \circ \phi$ is an isometry of $A(F)$ onto $A(E)$. Clearly ϕ need not be affine and therefore has no norm preserving extension in A. In this case

it is clear, however, that ϕ is generalized affine. And we prove next that such is the case in general.

We shall denote by m the second dual of A (that is, the dual of l^∞). For $\sigma \in m$ define

$$\hat{\sigma}(e^{it}) = < \{e^{int}\}, \sigma >.$$

An element $\sigma \in m$ is called an *extension of* a function f defined on some subset E of T, if

$$\hat{\sigma}(e^{it}) = f(e^{it}) \text{ for } e^{it} \in E.$$

LEMMA 3.1. *Let $f \in A(E)$. Then there exists in m an extension σ of f such that*

$$\|f\|_{A(E)} = \|\sigma\|_m.$$

Proof. f defines canonically a linear functional of norm $\|f\|_{A(E)}$ on the subspace $N(E)$ of l^∞. By the Hahn-Banach theorem this functional has a norm preserving extension $\sigma \in m$. Now if $e^{it} \in E$, $\{e^{int}\} \in N(E)$, and

$$f(e^{it}) = < \{e^{int}\}, f> = < \{e^{int}\}, \sigma> = \hat{\sigma}(e^{it}).$$

We turn now to the proof of Theorem 1.1. As in the proof of Proposition 2.2 we can assume $\phi(1) = 1$. Let $\sigma \in m$ be a norm preserving extension of ϕ, that is $\|\sigma\| = 1$, and $\hat{\sigma}(e^{it}) = \phi(e^{it})$ on E. We claim that $\hat{\sigma}(e^{it})$ is positive definite on T_D, T_D being the circle group with the discrete topology. In order to see this we can restrict σ to the closed subspace of l^∞ generated by the exponentials $\{\{e^{int}\}, e^{it} \in T\}$, i.e., the uniformly almost periodic sequences; as such, σ is a measure of mass 1 on B, the Bohr compactification of Z (and the dual of T_D). Since $\hat{\sigma}(1) = 1$, the measure corresponding to σ is positive and $\hat{\sigma}(e^{it})$ is positive definite. By Lemma 2.1 $\sigma(e^{it})$ is multiplicative on some subgroup G of T containing E; it is easy to see ([3] Th. 37) that $\hat{\sigma}(e^{it})|_G$ can be extended to be multiplicative on T, and ϕ is generalized affine as claimed.

Theorem 1.2 is an immediate consequence of Theorem 1.1 and of the following:

LEMMA 3.2. *Let E be closed in T and a basis, χ a character of T such that $\chi|_E$ is continuous. Then χ is continuous.*

Proof. Put $E_m = \{e^{it}: t = \sum_1^m t_j, e^{it_j} \in E\}$. Since E is a basis $T = \bigcup_m E_m$, and since E_m is closed for all m, we obtain, using the theorem of Baire, $T = E_m$ for $m > m_0$. In order to show that χ is continuous we have to show that, given $\varepsilon > 0$, there exists some interval I on T such that the variation of χ on I is smaller than ε. Put $\varepsilon_1 = \varepsilon/m_0$ and $E = \bigcup_{j=1}^k E^j$, E^j being portions of E on which the variation of χ is smaller than ε_1.

Clearly, for an appropriate choice of $j_1, \cdots, j_{m_0}$, $E' = \{e^{it} : t = \sum_1^{m_0} t_\nu, e^{it_\nu} \in E^{j_\nu}\}$ contains an interval I and the variation of χ on E' is smaller than $m_0\varepsilon_1 = \varepsilon$.

4. **Sufficient conditions for the existence of norm-preserving extensions.** Let E be closed in T. For $0 \leqq \varepsilon \leqq 1$ we denote by $N_\varepsilon(E)$ the following set:

$$(4.1) \qquad N_\varepsilon(E) = \{S : S \in N(E),\ \|S\| = 1,\ \limsup_{|n|\to\infty} |\hat{S}(n)| \leqq 1 - \varepsilon\}.$$

PROPOSITION 4.1. *Let $f \in A(E)$, $0 < \varepsilon \leqq 1$. Suppose that*

$$(4.2) \qquad \|f\|_{A(E)} = \sup |< f, S >|,\ S \in N_\varepsilon(E).$$

Then f has a norm preserving extension in A.

Proof. Let $\sigma \in m = (l^\infty)^*$ be a generalized norm preserving extension of f (cf. Lemma 3.1). Because of the canonical identification of l^∞ with the space of all continuous functions on the Čech compactification $\beta(Z)$ of Z, σ can be identified with a measure μ on $\beta(Z)$ and $\|\sigma\| = \int |d\mu|$. μ has a decomposition $\mu = \mu_1 + \mu_2$, with μ_1 concentrated on Z and μ_2 vanishing on every finite subset of Z, such that

$$(4.3) \qquad \int |d\mu| = \int |d\mu_1| + \int |d\mu_2|.$$

Denote by σ_j the elements in m corresponding to μ_j, so $\sigma = \sigma_1 + \sigma_2$ and

$$(4.4) \qquad \|\sigma\| = \|\sigma_1\| + \|\sigma_2\|.$$

Now for every $\{h(n)\} \in l^\infty$

$$(4.5) \qquad |< \{h(n)\}, \sigma_2 >| \leqq \|\sigma_2\| \cdot \limsup_{|n|\to\infty} |h(n)|.$$

So that if $S \in N_\varepsilon(E)$ we have

$$< f, S > = < S, \sigma > = < S, \sigma_1 > + < S, \sigma_2 >$$

and

$$(4.6) \qquad < f, S > | \leqq \|\sigma_1\| + (1 - \varepsilon)\|\sigma_2\| = \|f\|_{A(E)} - \varepsilon\|\sigma_2\|$$

which is compatible with (4.2) only if $\|\sigma_2\| = 0$. This means $\mu = \mu_1$ and hence $\sigma \in l^1$. We have shown therefore, not only that a norm preserving extension in A exists, but that every norm preserving extension is necessarily in A.

Proof of Theorem 1.5. Let $f \in A(E)$, $\|f\| = 1$, $|f(e^{it})| = 1$. Since f is continuous on E it is clear that if μ is a positive measure which has a very small support contained in E, and $\int d\mu = 1$, $|\int f d\mu|$ will be very close to $1 = \|f\|$. This remark and the assumptions of Theorem 1.5 imply (4.2), and the result follows from Proposition 4.1.

We conclude with the proof of Theorem 1.4. This can be done via Proposition 4.1 but it is almost as easy to give a direct proof, and that is what we do.

Proof of Theorem 1.4. We denote, as is usual, by c_0 the subspace of l^∞ of those sequences that converge to zero. Canonically $c_0^* = l^1$. Put $L(E) = N(E) \cap c_0$. If $f \in A(E)$, f defines a linear functional of norm $\|f\|'$ on $L(E)$, where clearly $\|f\|' \leqq \|f\|_{A(E)}$. By the Hahn-Banach theorem this functional can be extended to a functional of the same norm on c_0; there exists, therefore, an element $h \in l^1$, $\|h\| = \|f\|'$, such that for every $S \in L(E)$,

$$< f, S > = < S, h > = \sum \hat{S}(-n) h(n).$$

We contend that if E is UM^* we have $f(e^{it}) = \hat{h}(e^{it}) (= \sum h(n) e^{int})$ for $e^{it} \in E$. That is, $\hat{h}(e^{it})$ is the extension we seek. Assume $f \not\equiv \hat{h}$ on E. For some $e^{it_0} \in E$, $f(e^{it_0}) - \hat{h}(e^{it_0}) \neq 0$, and by continuity there exists an interval J containing e^{it_0} such that $f - \hat{h}$ is bounded away from zero on $J \cap E$. Let $S \neq 0$ be a pseudo-function carried by $J \cap E$. Let $g \in A$ such that $g(e^{it}) = [f(e^{it}) - \hat{h}(e^{it})]^{-1}$ for $e^{it} \in J \cap E$, and n be an arbitrary integer. $ge^{int} S \in L(E)$, so $< f - \hat{h}, ge^{int} S > = 0$. But

$$< f - \hat{h}, ge^{int} S > = < e^{int}, S > = \hat{S}(-n),$$

and consequently $\hat{S}(n) \equiv 0$ and $S = 0$, which gives the desired contradiction.

BIBLIOGRAPHY

1. A Beurling and H. Helson, *Fourier-Stieltjes transforms with bounded powers*, Math. Scand. **1** (1953), 120–126.

2. L. Loomis, *Introduction to abstract harmonic analysis*, Van Nostrand, 1953.

3. L. Pontrjagin, *Topological groups*, Princeton, 1946.

STANFORD UNIVERSITY
YALE UNIVERSITY